UDC 62.50

CONSTRUCTION OF A STABLE BRIDGE FOR A CLASS OF LINEAR GAMES

PMM Vol. 41, № 2, 1977, pp. 358-361

V. I. UKHOBOTOV
(Cheliabinsk)

(Received January 30, 1976)

A u-stable bridge (generally speaking not maximal) is constructed for a class of games. Sufficient conditions are found, under which the game in the given class can be completed within the time of the first instant of absorption.

1. A vector z moves in an n-dimensional Euclidean space R^n according to the following equation:

$$z^{\cdot} = Cz - u + v, \quad u \in P, \quad v \in Q \tag{1.1}$$

where C is a constant ($n \times n$)-matrix, while P and Q are convex compacts in R^n. It is assumed that an m-dimensional Euclidean space R^m $(m \leqslant n)$, a linear mapping π of the space R^n onto R^m and a closed convex set $X \subset R^m$, are all specified. A terminal set Z is defined within R^n as follows:

$$Z = \{z \in R^n : \pi z \in X\} \tag{1.2}$$

We consider the problem of hitting the vector z on the terminal set Z at the given instant of time t_1. The first player who has the choice of the control $u \in P$ at his disposal, tries to realize this action, while the second player using the control $v \in Q$ obstructs the intention of the first player. The strategies of the players are identified with the functions $u(t, z) \in P$, $v(t, z) \in Q$ (see [1]). It was shown in [1] that the strategy of the first player extremal to the u-stable bridge W leading to the target Z, ensures that the vector z hits the set Z.

We shall seek the u-stable bridge W in the following form:

$$W = \{(\tau, z): \ 0 \leqslant \tau \leqslant t_1, \ \pi e^{\tau C} z \in T(\tau)\} \tag{1.3}$$

The family of sets $T(\tau) \subset R^m$ $(0 \leqslant \tau \leqslant t_1)$ is obtained from the property of the u-stability of the bridge W and the condition $T(0) = X$. The bridge (1.3) will be u-stable (see [1]) when the following condition holds: if $(\tau, z_0) \in W$ and a measurable control $v(t) \in Q$ is chosen for $0 \leqslant t \leqslant \sigma$ $(\sigma \leqslant \tau)$, then a measurable control $u(t) \in P$ $(0 \leqslant t \leqslant \sigma)$ exists for which $(\tau - \sigma, z(\sigma)) \in W$. Here $z(\sigma)$ denotes the position of the vector z to which the latter is displaced from its initial position z_0 at the instant σ, in accordance with (1.1), under the controls $u(t)$ and $v(t)$ As we know [2], to fulfil this condition it is sufficient that the following inclusion holds:

$$T(\tau) \subset \left(T(\tau - \sigma) + \int_{\tau-\sigma}^{\tau} \pi e^{tC} P \, dt\right) \overset{*}{-} \left(\int_{\tau-\sigma}^{\tau} \pi e^{tC} Q \, dt\right) \tag{1.4}$$

Here $A \overset{*}{-} B = \bigcap (A - b)$ $(b \in B)$ is the geometrical difference (see [2]) of two sets, $A \subset R^m$ and $B \subset R^m$.

2. Let us now construct a family of sets $T(\tau)$ satisfying the inclusion (1.4) and condition $T(0) = X$, under the assumption that the following condition holds:

$$\pi e^{tC}P = y_1(t) + k_1(t)\,S_1, \quad \pi e^{tC}Q = y_2(t) + k_2(t)\,S_2 \tag{2.1}$$

Here S_i denote convex compacts in R^m, $k_i(t)$ are continuous scalar functions, $k_i(t) \geqslant 0$ when $t \geqslant 0$, and $y_i(t)$ are continuous m-dimensional vector functions; $i = 1, 2$. We seek the set $T(\tau)$ in the form

$$T(\tau) = \int_0^\tau (y_1(t) - y_2(t))\,dt + W(\tau) \tag{2.2}$$

Substituting the set (2.2) into (1.4) and utilizing the condition (2.1), we obtain an inclusion which must be satisfied by the set $W(\tau)$

$$W(\tau) \subset (W(\tau - \sigma) + \Big(\int_{\tau-\sigma}^{\tau} k_1(t)\,dt\,S_1\Big) \overset{*}{-} \Big(\Big(\int_{\tau-\sigma}^{\tau} k_2(t)\,dt\Big) S_2\Big) \tag{2.3}$$

In what follows, we shall require certain properties of the geometrical difference operation. Let A be a closed convex set in R^m, B and K be convex compacts in R^m, σ_1 and σ_2 be nonnegative numbers. Then

$$(A \overset{*}{-} B) \overset{*}{-} K = A \overset{*}{-} (B + K) \tag{2.4}$$

$$(A + B) \overset{*}{-} K \supset (A \overset{*}{-} K) + B \tag{2.5}$$

$$(((A + \sigma_1 B) \overset{*}{-} \sigma_1 K) + \sigma_2 B) \overset{*}{-} \sigma_2 K = (A + (\sigma_1 + \sigma_2) B) \overset{*}{-} (\sigma_1 + \sigma_2) K \tag{2.6}$$

$$(A + B) \overset{*}{-} (K + B) = A \overset{*}{-} K \tag{2.7}$$

The proof of the properties (2.4) and (2.5) and of the inclusion $(A + B) \overset{*}{-} (K + B) \supset A \overset{*}{-} K$ for arbitrary sets A, B and K is given in [3], p. 204. The property (2.6) is shown in [4, 5] in the course of the proof of the equation $T_{\sigma_1} T_{\sigma_2} = T_{\sigma_1+\sigma_2}$ for the games with a simple motion and a convex terminal set. The proof in [4, 5] utilizes the concept of a supporting function of a convex closed set. Using this concept, we can also easily prove the property (2.7).

The following lemma generalizes the property (2.6).

L e m m a. Let the numbers $p_1 \geqslant 0$, $p_2 \geqslant 0$, $\delta_1 \geqslant 0$, $\delta_2 \geqslant 0$ be such, that

$$p_2\delta_1 - p_1\delta_2 \geqslant 0 \tag{2.8}$$

Then the following relation holds:

$$(((A + p_2 B) \overset{*}{-} \delta_2 K) + p_1 B) \overset{*}{-} \delta_1 K = (A + (p_1 + p_2) B) \overset{*}{-} (\delta_1 + \delta_2) K \tag{2.9}$$

P r o o f : Let us first consider the case when $p_1 = 0$. Then the relation (2.9) follows from the property (2.4) and the relation $(\delta_1 + \delta_2) K = \delta_1 K + \delta_2 K$.

Let $p_1 > 0$. Then by virtue of the condition (2.8) we can determine the nonnegative number

$$\varepsilon = (p_2\delta_1 - p_1\delta_2) / p_1 \tag{2.10}$$

We can also assume that in the case in question $(p_1 > 0)$ we have $p_2 > 0$ otherwise the condition (2.8) would imply that $p_2 = \delta_2 = 0$ and the relation (2.9) would become obvious. Let us set $A_1 = A + \varepsilon K$. Then the property (2.7) implies the relation

$$(A + p_2 B) \overset{*}{-} \delta_2 K = (A_1 + p_2 B) \overset{*}{-} (\delta_2 + \varepsilon) K \tag{2.11}$$

Choosing the value of the number ε from (2.10) we obtain

$$\delta_1 / p_1 = (\delta_2 + \varepsilon) / p_2 = \alpha \geqslant 0 \tag{2.12}$$

Let us now set $K_1 = \alpha K$. Then, taking into account (2.12) we obtain $(\delta_2 + \varepsilon) K = p_2 K_1$,

$\delta_1 K = p_1 K_1$. From this, together with (2.11), follows that the left-hand side of the relation (2.9) which is being proved, has the form

$$(((A_1 + p_2 B) \overset{*}{-} p_2 K_1) + p_1 B) \overset{*}{-} p_1 K_1$$

According to the property (2.6), the above set is equal to

$$(A_1 + (p_1 + p_2) B) \overset{*}{-} (p_1 + p_2) K_1$$

Substituting in it $A_1 = A + \varepsilon K$, $(p_1 + p_2) K_1 = (\delta_1 + \delta_2) K + \varepsilon K$ and using the property (2.7), we obtain (2.9).

3. Let us now determine the family of sets $W(\tau)$ satisfying the inclusion (2.3) and condition $W(0) = X$.

We determine the function $f(\tau)$ for $\tau \geqslant 0$ as a solution of the following differential equation:

$$\frac{df(\tau)}{d\tau} = \max\left\{k_2(\tau);\ \frac{f(\tau)\, k_1(\tau)}{J_1(0,\tau)}\right\}, \quad f(0) = 0 \tag{3.1}$$

Here

$$J_\alpha(a, b) = \int_a^b k_\alpha(t)\, dt, \quad \alpha = 1, 2$$

We note that if $J_1(0, \tau) = 0$ for some $\tau > 0$, then $k_1(t) = 0$ for $0 \leqslant t \leqslant \tau$. This, together with condition (2.1), implies that for $0 \leqslant t \leqslant \tau$ and for any $u_1 \in P$, $u_2 \in P$, the following relation holds:

$$\pi e^{tC} u_1 = \pi e^{tC} u_2 \tag{3.2}$$

Representing the matrix e^{tC} by a power series and making t tend to zero, we obtain $\pi C^i u_1 = \pi C^i u_2$ for all t. It follows that the equality (3.2) holds for all t, and we can set $k_1(t) = 0$ in the first equation of (2.1) for all $t \geqslant 0$. In this case we assume that the right-hand side of (3.1) is equal to $k_2(t)$.

Equation (3.1) yields the following properties of the function $f(\tau)$:

$$f(\tau) \geqslant J_2(0, \tau) \geqslant 0,\ f(\tau) - f(\tau - \sigma) \geqslant J_2(\tau - \sigma, \tau) \geqslant 0 \tag{3.3}$$

$$J_1(0, \tau - \sigma)\, f(\tau) \geqslant J_1(0, \tau)\, f(\tau - \sigma),\ 0 \leqslant \sigma \leqslant \tau \tag{3.4}$$

We shall prove the inequality (3.4). Set

$$x(\tau) = f(\tau) / J_1(0, \tau)$$

Then as (3.1) implies, $dx(\tau) / d\tau \geqslant 0$ when $\tau > 0$. This means that for $0 \leqslant \sigma \leqslant \tau$, $x(\tau) \geqslant x(\tau - \sigma)$, and taking into account the form of the function $x(\tau)$ we obtain the inequality (3.4).

Let us set for every $\tau \geqslant 0$

$$M(\tau) = (X + J_1(0, \tau) S_1) \overset{*}{-} f(\tau) S_2 \tag{3.5}$$

We note that the set $M(\tau)$, being the geometrical difference of two convex sets, is itself convex.

Assertion. Let the set $M(\tau)$ be nonempty for every $\tau \in [0, t_1]$. Then for $\tau \in [0, t_1]$ the inclusion (2.3) is satisfied by the following family of sets:

$$W(\tau) = M(\tau) + (f(\tau) - J_2(0,\tau)) S_2 \tag{3.6}$$

Proof. From the inequality (3.4) it follows that

$$(f(\tau) - f(\tau - \sigma))\, J_1(0, \tau - \sigma) \geqslant f(\tau - \sigma)\, J_1(\tau - \sigma, \tau)$$

This, together with the formula (3.5) and the lemma proved above, yields the relation

$$(M(\tau-\sigma)+J_1(\tau-\sigma,\tau)S_1) \overset{*}{-} (f(\tau)-f(\tau-\sigma))S_2 = M(\tau) \qquad (3.7)$$

We shall show that

$$(M(\tau-\sigma)+J_1(\tau-\sigma,\tau)S_1) \overset{*}{-} J_2(\tau-\sigma,\tau)S_2 \supset M(\tau)+\varepsilon_1 S_2$$

where

$$\varepsilon_1 = f(\tau)-f(\tau-\sigma)-J_2(\tau-\sigma,\tau) \geqslant 0 \qquad (3.9)$$

We note by I the set appearing in the left-hand side of the inclusion (3.8) which is being proved. Then from property (2.7) and from (3.9) it follows that

$$I = (M(\tau-\sigma)+J_1(\tau-\sigma,\tau)S_1+\varepsilon_1 S_2) \overset{*}{-} (f(\tau)-f(r-\sigma))S_2$$

Applying to this the property (2.5) and from the relation (3.7), we obtain the inclusion (3.8).

Let us substitute the set (3.6) into the right-hand side of the inclusion (2.3) which is being proved, and denote the resulting set by I_1. According to (2.5), we then have the following inclusion:

$$I_1 \supset ((M(\tau-\sigma)+J_1(\tau-\sigma,\tau)S_1) \overset{*}{-} J_2(\tau-\sigma,\tau)S_2)+(f(\tau-\sigma)-J_2(0,\tau-\sigma))S_2$$

The latter yields, with the help of the relation (3.8) and the notation (3.9), the required inclusion $I_1 \supset W(\tau)$.

The above assertion together with the formulas (1.3) and (2.2) implies that starting from the initial position z_0, the first player will be able to realize the hit of the vector z on the terminal set (1.2) at the time $t_1 > 0$, provided that

$$\pi e^{t_1 C} z_0 \in \int_0^{t_1} (y_1(t)-y_2(t))\,dt + M(t_1)+(f(t_1)-J_2(0,t_1)S_2 \qquad (3.10)$$

The author of [6] introduced the concept of the first instant of absorption. We know [7] that in a number of cases the game can be concluded by the time equal to the first instant of absorption. We shall give the sufficient conditions under which the games belonging to the class in question can be concluded in the time coinciding with the first instant of absorption.

When the condition (2.1) holds and the first instant of absorption $t_1 = t_1(z_0)$ exists, the following inclusion holds:

$$\pi e^{t_1 C} z_0 \in \int_0^{t_1} (y_1(t)-y_2(t))\,dt + (X+J_1(0,t_1)S_1) \overset{*}{-} J_2(0,t_1)S_2 \qquad (3.11)$$

From the inclusions (3.10) and (3.11) we see that the sufficient conditions sought can be found by requiring that the sets appearing in the right-hand sides of these inclusions are equal. As the formula (3.5) implies, to achieve this it is sufficient to require that $f(\tau) = J_2(0,\tau)$. The latter equality holds if, as we see from (3.1), the following inequality holds:

$$k_2(\tau)\int_0^{\tau} k_1(t)\,dt \geqslant k_1(\tau)\int_0^{\tau} k_2(t)\,dt, \quad \tau \geqslant 0 \qquad (3.12)$$

Thus if the terminal set has the form (1.2), the condition (2.1) and the inequality (3.12) both hold, then the game can be concluded by the time equal to the first instant of absorption.

REFERENCES

1. Krasovskii, N. N. and Subbotin, A. I., Positional Differential Games. Moscow, "Nauka", 1974.

2. Pontriagin, L. S., On linear differential games. 2. Dokl. Akad. Nauk SSSR, Vol. 175, № 4, 1967.

3. Khadviger, G., Lectures on Volume, Area of Surface and Isoperimetry. Moscow, "Nauka", 1966.

4. Pshenichnyi, B. N., Game with a simple motion and a convex terminal set. Tr. Inst. kibernetiki, № 3, 1969.

5. Pshenichnyi, B. N. and Sagaidak, M. I., On the differential games with fixed time. Kibernetika, № 2, 1970.

6. Krasovskii, N. N., On a problem of tracking. PMM Vol. 27, № 2, 1963.

7. Krasovskii, N. N., Game Problems of Encounter of Motion. Moscow, "Nauka", 1970.

Translated by L. K.